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On the maximization of (not necessarily) convex functions on convex sets

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Abstract The global solutions of the problem of maximizing a convex function on a convex set were characterized by several authors using the Fenchel (approximate) subdifferential. When the objective function is quasiconvex it was considered the differentiable case or used the Clarke subdifferential. The aim of the present paper is to give necessary and sufficient optimality conditions using several subdifferentials adequate for quasiconvex functions. In this way we recover almost all the previous results related to such global maximization problems with simple proofs.

Keywords Global maximization · Normal cone · Quasiconvex function · Quasi relative interior · Subdifferential

1 Introduction

In this note we are concerned with necessary and sufficient conditions of optimality for the maximization problem

$$(P) \qquad \text{maximize } f(x) \text{ subject to } x \in C,$$

where f is a proper extended valued function defined on the separated locally convex space X and $C \subset X$ is a nonempty set. We are interested mainly in the case when f is convex, possibly in a generalized sense, and C is (mainly) a convex set. The first result related to (P) when f and C are convex can be found in Rockafellar's book [15]. As mentioned in [15, Section 32], "the theory of the maximum of a convex function relative to a convex set has an entirely different character from the theory of the minimum". The difference between the minimization and the maximization of a convex function on a convex set can be seen from the optimality conditions. So, in the case $\dim X < \infty$, $\overline{x} \in C \cap \operatorname{ri}(\operatorname{dom} f)$ is a solution for

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the problem of minimizing f on C if and only if $\partial f(\overline{x}) \cap -N(C,\overline{x}) \neq \emptyset$, while in the same situation the necessary condition for $\overline{x}$ to be a solution of (P) is

$$\partial f(\overline{x}) \subset N(C,\overline{x}).$$

Note that this condition is not sufficient for $\overline{x}$ to be a solution of (P); take for example $X := \mathbb{R}^2$, $f(x_1,x_2) := x_1^2 + x_2^2$, $\overline{x} := (1,0)$ and $C := \{1\} \times \mathbb{R}$ (then $\partial f(\overline{x}) = \{(2,0)\}$, $N(C,\overline{x}) = \mathbb{R} \times \{0\}$ and $f(x) > f(\overline{x})$ for every $x \in C \setminus \{\overline{x}\}$). Hirriart-Urruty [9] considered the problem (P) when $f : X \to \mathbb{R}$ is a continuous convex function and C is a closed convex set as a particular case of d.c. programming problems and he provided a necessary and sufficient condition for the optimality of $\overline{x} \in C$:

$$\partial_\varepsilon f(\overline{x}) \subset N_\varepsilon(C,\overline{x}) \quad \forall \varepsilon \geq 0.$$

When f is convex, under the supplementary condition $\inf_{x\in X} f(x) < f(\overline{x})$, f is continuous on int(dom f) and $[f \leq f(\overline{x})] \subset$ int(dom f), Strekalovski [19] provided the following characterization for the optimality of $\overline{x} \in C$:

$$\partial f(x) \subset N(C,x) \quad \forall x \in X \quad \text{with } f(x) = f(\overline{x}).$$

(Note that Strekalovski [17] established another characterization of a solution of (P) we do not discuss here.)

The problem of finding necessary or sufficient optimality conditions for (P) was further studied by several authors: Hirriart-Urruty and Ledyaev [10] for locally Lipschitz functions, Dür et al. [4] for the classical case, Tsevendorj [20] for convex and piecewise-convex functions, Enkhbat et al. [6], Enkhbat and Ibaraki [7] for quasiconvex differentiable functions, and recently by Dutta [3] for generalized convex locally Lipschitz functions.

Our aim is to provide some new optimality conditions for problem (P) using normal cone type subdifferentials and to derive (several times in more general conditions), with simple proofs, almost all known results on this subject.

2 Optimality conditions in the quasiconvex case

Throughout the paper X is a (Hausdorff) locally convex space whose topological dual is denoted by X^*, $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is a proper function (that is, f is finite somewhere and does not take the value $-\infty$), $C \subset \text{dom } f := \{x \in X \mid f(x) < \infty\}$ and $\overline{x} \in C$. For $t \in \mathbb{R}$ we set $[f \leq t] := \{x \in X \mid f(x) \leq t\}$, and similarly for $[f = t]$, $[f < t]$. Of course, $\overline{x}$ is a *solution of* (P) if $f(\overline{x}) \geq f(x)$ for every $x \in C$. It is clear that

$$\overline{x} \text{ is solution of } (P) \quad \Leftrightarrow \quad C \subset [f \leq f(\overline{x})]. \tag{1}$$

Defining, as usual, the ε-*normal set* and the *normal cone* of the nonempty set $A \subset X$ at $a \in X$ by

$$N_\varepsilon(A,a) := \{x^* \in X^* \mid \langle x - a, x^*\rangle \leq \varepsilon \ \forall x \in A\}, \quad N(A,a) := N_0(A,a),$$

the equivalence (1) yields immediately optimality conditions involving some subdifferentials used in quasiconvex analysis. Recall that the *Greenberg–Pierskalla subdifferential* of f at $\overline{x} \in \text{dom } f$ is the set

$$\partial^* f(\overline{x}) := \{x^* \in X^* \mid \langle x - \overline{x}, x^*\rangle < 0 \ \forall x \in [f < f(\overline{x})]\};$$

so, if $\overline{x}$ is a global minimum of f then $\partial^* f(\overline{x}) = X^*$. This is a subdifferential of normal type; other such subdifferentials are

$$\partial^\nu f(\overline{x}) := \{x^* \in X^* | \langle x - \overline{x}, x^* \rangle \leq 0 \, \forall x \in [f \leq f(\overline{x})]\} = N([f \leq f(\overline{x})], \overline{x}),$$
$$\partial^\circledast f(\overline{x}) := \{x^* \in X^* | \langle x - \overline{x}, x^* \rangle \leq 0 \, \forall x \in [f < f(\overline{x})]\} = N([f < f(\overline{x})], \overline{x})$$

(see [13]). We also consider the approximate subdifferential

$$\partial^\nu_\varepsilon f(\overline{x}) := N_\varepsilon([f \leq f(\overline{x})], \overline{x})$$

for $\varepsilon \geq 0$. In quasiconvex analysis one uses also the *lower subdifferential* of Plastria [14] and the *infradifferential* of Gutiérrez [8] which are defined by

$$\partial^< f(\overline{x}) := \{x^* \in X^* | \langle x - \overline{x}, x^* \rangle \leq f(x) - f(\overline{x}) \, \forall x \in [f < f(\overline{x})]\},$$
$$\partial^\leq f(\overline{x}) := \{x^* \in X^* | \langle x - \overline{x}, x^* \rangle \leq f(x) - f(\overline{x}) \, \forall x \in [f \leq f(\overline{x})]\},$$

respectively. Obviously, one has

$$\partial^\leq f(\overline{x}) \subset \partial^< f(\overline{x}) \subset \partial^* f(\overline{x}), \quad \partial^\leq f(\overline{x}) \subset \partial^\nu f(\overline{x}).$$

As we shall see below, using $\partial^* f$ and $\partial^\nu f$ is more exact in our context. Recall that f is *quasiconvex* when $[f \leq t]$ is convex for every $t \in \mathbb{R}$.

Proposition 1 *If $\overline{x}$ is a solution of (P) then*

$$\partial^\nu_\varepsilon f(\overline{x}) \subset N_\varepsilon(C, \overline{x}) \quad \forall \varepsilon \geq 0. \tag{2}$$

Moreover, if $[f \leq f(\overline{x})]$ is a closed convex set (in particular if f is a lsc quasi-convex function) and (2) holds, then $\overline{x}$ is a solution of (P).

Proof Assuming that $\overline{x}$ is a solution of (P), we have $C \subset [f \leq f(\overline{x})]$, whence (2) holds.

Assume now that (2) holds and $[f \leq f(\overline{x})]$ is a closed convex set. We have to show $C \subset [f \leq f(\overline{x})]$. Consider $x \in X \backslash [f \leq f(\overline{x})]$. By a separation theorem, there exists $x^* \in X^*$ such that $\langle x, x^* \rangle > \sup\{\langle u, x^* \rangle | u \in [f \leq f(\overline{x})]\} \geq \langle \overline{x}, x^* \rangle$. Consider $\varepsilon := \sup\{\langle u - \overline{x}, x^* \rangle \mid u \in [f \leq f(\overline{x})]\} \geq 0$. Then $x^* \in \partial^\nu_\varepsilon f(\overline{x})$, and so $x^* \in N_\varepsilon(C, \overline{x})$. Since $\langle x - \overline{x}, x^* \rangle > \varepsilon$, it follows that $x \notin C$. Hence $C \subset [f \leq f(\overline{x})]$. □

The result stated in Proposition 1 is the quasiconvex version of the well-known characterization of optimality of $\overline{x}$ for (P) given by Hirriart-Urruty [9]. We can deduce Hirriart-Urruty's result from Proposition 1. For this we need the next result which generalizes [12, Prop. 5.4] in which $\Lambda(X)$ denotes the class of proper convex functions on X and

$$\partial_\varepsilon g(x) := \{x^* \in X^* \mid \langle y - x, x^* \rangle \leq g(y) - g(x) + \varepsilon \forall y \in X\}, \quad \partial g(x) := \partial_0 g(x)$$

are the ε- and *Fenchel subdifferentials* of the proper function $g \colon X \to \overline{\mathbb{R}}$ at $x \in \operatorname{dom} g$, with $\partial_\varepsilon g(x) := \partial g(x) := \emptyset$ for $x \notin \operatorname{dom} g$; moreover, the *conjugate* of g is the function

$$g^* : X^* \to \overline{\mathbb{R}}, \quad g^*(x^*) := \sup\{\langle x, x^* \rangle - g(x) \mid x \in X\}.$$

Proposition 2 *Let $g \in \Lambda(X)$, $\overline{x} \in \operatorname{dom} g$ be such that $[g < g(\overline{x})] \neq \emptyset$, and $\varepsilon \geq 0$. Then*

$$\partial^\nu_\varepsilon g(\overline{x}) = N_\varepsilon(\operatorname{dom} g, \overline{x}) \cup \bigcup_{\lambda > 0} \lambda \partial_{\lambda^{-1}\varepsilon} g(\overline{x}) = \bigcup_{\lambda \geq 0} \partial_\varepsilon(\lambda g)(\overline{x}).$$

Proof The inclusion ⊃ is obtained by an easy calculation. For the converse take $x^* \in \partial_\varepsilon^\nu g(\overline{x})$. This means that

$$\mu := \sup\{\langle x, x^*\rangle | x \in [g \le g(\overline{x})]\} \le \langle \overline{x}, x^*\rangle + \varepsilon. \tag{3}$$

Applying [12, Lemma 5.1] with $\gamma := g(\overline{x})$, we have $\mu = \sup\{\langle x, x^*\rangle | x \in \mathrm{dom}\, g\}$, whence $x^* \in N_\varepsilon(\mathrm{dom}\, g, \overline{x})$, or there exists $\lambda > 0$ such that $\mu = \lambda g(\overline{x}) + \lambda g^*(\lambda^{-1}x^*)$. From (3) we obtain $g(\overline{x}) + g^*(\lambda^{-1}x^*) \le \langle \overline{x}, \lambda^{-1}x^*\rangle + \lambda^{-1}\varepsilon$ which means that $\lambda^{-1}x^* \in \partial_{\lambda^{-1}\varepsilon} g(\overline{x})$, that is, $x^* \in \lambda \partial_{\lambda^{-1}\varepsilon} g(\overline{x})$. The conclusion follows. □

In the next result $\Gamma(X)$ denotes the class of proper convex and lower semicontinuous functions on X.

Proposition 3 *Let $f \in \Gamma(X)$ and assume that $[f < f(\overline{x})]$ is nonempty. Then $\overline{x}$ is a solution of (P) if and only if*

$$\partial_\varepsilon f(\overline{x}) \subset N_\varepsilon(C, \overline{x}) \quad \forall \varepsilon \ge 0. \tag{4}$$

Proof The necessity follows immediately from Proposition 1 (even for arbitrary f) because $\partial_\varepsilon f(\overline{x}) \subset \partial_\varepsilon^\nu f(\overline{x})$ for $\varepsilon \ge 0$. Taking into account that $[f \le f(\overline{x})]$ is a closed convex set, by Proposition 1, for the converse it is sufficient to show that (2) is verified. Indeed, take $x^* \in \partial_\varepsilon^\nu f(\overline{x})$. From Proposition 2 we have either $x^* \in N_\varepsilon(\mathrm{dom}\, f, \overline{x})$, and so $x^* \in N_\varepsilon(C, \overline{x})$ (since $C \subset \mathrm{dom}\, f$), or $x^* \in \lambda \partial_{\lambda^{-1}\varepsilon} f(\overline{x})$ for some $\lambda > 0$; then $\lambda^{-1}x^* \in \partial_{\lambda^{-1}\varepsilon} f(\overline{x}) \subset N_{\lambda^{-1}\varepsilon}(C, \overline{x}) = \lambda^{-1} N_\varepsilon(C, \overline{x})$. The conclusion follows. □

Remark 1 *Note that the condition $[f < f(\overline{x})] \ne \emptyset$ is superfluous in the previous result as a direct proof (of the sufficiency of (4)) shows:*

Assume that $f(x) > f(\overline{x})$ for some $x \in C$. Because $f \in \Gamma(X)$, there exists $x^* \in X^*$ such that $\langle x, x^*\rangle - f^*(x^*) > f(\overline{x})$. Let $\gamma \in \mathbb{R}$ be such that $f(\overline{x}) + f^*(x^*) < \gamma < \langle x, x^*\rangle$ and take $\varepsilon := \gamma - \langle \overline{x}, x^*\rangle > 0$. On one hand we have $\langle x - \overline{x}, x^*\rangle > \varepsilon$, which means that $x^* \notin N_\varepsilon(C, \overline{x})$. On the other hand, $f(\overline{x}) + f^*(x^*) < \gamma = \langle \overline{x}, x^*\rangle + \varepsilon$, which shows that $x^* \in \partial_\varepsilon f(\overline{x})$. This contradiction proves that $\overline{x}$ is a solution of (P).

When $\mathrm{dom}\, f = X$, f is convex and continuous and C is a closed convex set, the result mentioned in the previous remark was obtained by Hirriart-Urruty [9, Cor. 4.5].

When f is convex Strekalovski [18, 19] showed that under condition $[f < f(\overline{x}] \ne \emptyset$ it is possible to use only the subdifferential of f and the normal cone of C, but involving all the elements of $[f = f(\overline{x})]$. Our corresponding result is the following.

Proposition 4 (i) *If $\overline{x}$ is a solution of (P) then*

$$\partial^\nu f(x) \subset N(C, x) \qquad \forall x \in [f = f(\overline{x})]. \tag{5}$$

(ii) *Assume that $[f < f(\overline{x})] \ne \emptyset$,* dom *f is convex, and the restriction of f at any segment included in* dom *f is continuous. Then $\overline{x}$ is a solution of (P) provided*

$$\partial^* f(x) \cap N(C, x) \ne \emptyset \qquad \forall x \in [f = f(\overline{x})]. \tag{6}$$

Proof (i) If $\overline{x}$ is a solution of (P), then $x \in [f = f(\overline{x})]$ is also solution of (P) in which C is replaced by $C \cup \{x\}$; using Proposition 1 we get $\partial^\nu f(x) = \partial_0^\nu f(x) \subset N_0(C, x) = N(C, x)$.

(ii) Assume that $\overline{x}$ is not a solution of (P). Then there exists some $x_1 \in C$ such that $f(x_1) > f(\overline{x})$; since $C \subset \mathrm{dom}\, f$ we have $x_1 \in \mathrm{dom}\, f$. By hypothesis there exists $x_0 \in X$ such that $f(x_0) < f(\overline{x})$. Since $\mathrm{dom}\, f$ is convex, the segment $[x_0, x_1]$ is included in $\mathrm{dom}\, f$.

Moreover, because the restriction $f|_{[x_0,x_1]}$ of the function f to the segment $[x_0, x_1]$ is continuous and $f(x_0) < f(\overline{x}) < f(x_1)$, there exists $x := (1-\lambda)x_0 + \lambda x_1$ with $\lambda \in (0,1)$ such that $f(x) = f(\overline{x})$. By (6), there exists $x^* \in \partial^* f(x) \cap N(C,x)$. Then $\langle x_0 - x, x^* \rangle = \lambda \langle x_0 - x_1, x^* \rangle < 0$ because $x_0 \in [f < f(\overline{x})]$ and $x^* \in \partial^* f(x)$, and $\langle x_1 - x, x^* \rangle = (1-\lambda)\langle x_1 - x_0, x^* \rangle \leq 0$ because $x_1 \in C$ and $x^* \in N(C,x)$. We obtained so the contradiction $\langle x_0 - x_1, x^* \rangle < 0 \leq \langle x_0 - x_1, x^* \rangle$. □

The fact that in (5) and (6) there are involved points outside C is unnatural (as observed in [10, p. 56]). When C is convex we can avoid such a situation.

Proposition 5 (i) If $\overline{x}$ is a solution of (P) then

$$\partial^\nu f(x) \subset N(C,x) \quad \forall x \in C \cap [f = f(\overline{x})]. \tag{7}$$

(ii) Assume that C is convex, $C \cap [f < f(\overline{x})] \neq \emptyset$ and the restriction of f at any segment included in C is continuous. Then $\overline{x}$ is a solution of (P) provided

$$\partial^* f(x) \cap N(C,x) \neq \emptyset \quad \forall x \in C \cap [f = f(\overline{x})]. \tag{8}$$

Proof Assertion (i) is a particular case of Proposition 4(i). The proof of (ii) is the same as that of (ii) of Proposition 4 observing that we can take $x_0 \in C$, and so $[x_0, x_1] \subset C$. □

It is natural to ask about the relation between the conditions $\partial^\nu f(x) \subset N(C,x)$ (or its equivalent condition $\partial^\nu f(x) \backslash \{0\} \subset N(C,x)$ and $\partial^* f(x) \cap N(C,x) \neq \emptyset$. The next result from [13] is useful in this sense, where *f radially usc at x* means that the restriction of f at any line passing through x is usc at x.

Proposition 6 [13, Lemma 2.1]

(i) *If $\partial^* f(\overline{x})$ is nonempty, then $\partial^{\circledast} f(\overline{x}) = \mathrm{cl}\,\partial^* f(\overline{x})$.*
(ii) *If f is radially usc at each point of $[f < f(\overline{x})]$ then $\partial^{\circledast} f(\overline{x}) = \partial^* f(\overline{x}) \cup \{0\}$.*
(iii) *If there is no local minimizer of f in $[f = f(\overline{x})]$ then $\partial^{\circledast} f(\overline{x}) = \partial^\nu f(\overline{x})$.*

The first application of the preceding results is the next extension of Theorem 2.1 in Enkhbat and Ibaraki's paper [7] (stated in the case $X = \mathbb{R}^n = \mathrm{dom}\, f$); see also [19, p. 351].

Proposition 7 *Assume that f is a quasiconvex function which is Gâteaux differentiable on* dom f. *If $\overline{x}$ is a solution of (P) then*

$$\nabla f(x) \in N(C,x) \quad \forall x \in [f = f(\overline{x})]. \tag{9}$$

Conversely, $\overline{x}$ is a solution of (P) provided

$$\nabla f(x) \in N(C,x) \backslash \{0\} \quad \forall x \in [f = f(\overline{x})]. \tag{10}$$

Proof Observe first that since f is Gâteaux differentiable on dom f, dom f coincides with its algebraic interior and f is radially continuous on dom f.

Because f is quasiconvex, for $x \in \mathrm{dom}\, f$ and $y \in [f \leq f(x)]$ we have $f(x + t(y-x)) \leq f(x)$ for every $t \in (0,1)$, and so $\nabla f(x)(y-x) = \lim_{t \downarrow 0} t^{-1}[f(x+t(y-x)) - f(x)] \leq 0$. This proves that $\nabla f(x) \in \partial^\nu f(x)$, whence (9) follows using Proposition 4(i).

Assume that (10) holds. Because $\nabla f(x) \neq 0$ for every $x \in [f = f(\overline{x})]$, f has no local minimizers in $[f = f(\overline{x})]$. Since f is radially (upper semi-) continuous on dom f, by the preceding proposition we have $\partial^\nu f(x) = \partial^* f(\overline{x}) \cup \{0\}$, and so $\nabla f(x) \in \partial^* f(\overline{x})$ for every $x \in [f = f(\overline{x})]$. The conclusion follows using Proposition 4(ii). □

When f is radially usc at each point of $[f < f(\overline{x})]$ and f has not local minimizers in $[f = f(\overline{x})]$, from Proposition 6, we observe that (5) is equivalent to

$$\partial^* f(x) \subset N(C, x) \qquad \forall x \in [f = f(\overline{x})]$$

and (7) is equivalent to

$$\partial^* f(x) \subset N(C, x) \qquad \forall x \in C \cap [f = f(\overline{x})]. \tag{11}$$

So, when $\partial^* f(x) \neq \emptyset$ for every $x \in [f = f(\overline{x})]$ (resp. for every $x \in C \cap [f = f(\overline{x})]$) condition (6) (resp. (8)) is weaker than condition (5) (resp. (7)).

The next simple criterion for the nonemptiness of $\partial^* f(x)$ can be found in Penot [11, p. 65]: *Assume that f is quasiconvex,* $f(x) \in \mathbb{R}$ *and f is upper semicontinuous at any point in* $[f < f(x)]$*. Then* $\partial^* f(x) \neq \emptyset$*.*

Proposition 5 together with the discussion above yield the next result.

Proposition 8 *Assume that f is quasiconvex and continuous on* dom f *and has not local minimizers on* $[f = f(\overline{x})]$*, C is convex and* $C \cap [f < f(\overline{x})] \neq \emptyset$*. Then "*$\overline{x}$ *is a solution of* (P)*" is characterized by anyone of the conditions* (7)*,* (8) *and* (11)*.*

Because obviously $\partial f(x) \subset \partial^\nu f(x)$ for every $x \in \text{dom } f$, the conditions

$$\partial f(x) \subset N(C, x) \qquad \forall x \in [f = f(\overline{x})] \tag{12}$$

and

$$\partial f(x) \subset N(C, x) \qquad \forall x \in C \cap [f = f(\overline{x})] \tag{13}$$

are also necessary conditions for the optimality of $\overline{x}$. Since $\partial f(x) \subset \partial^* f(x)$ for every $x \in \text{dom } f$, the conditions

$$\partial f(x) \cap N(C, x) \neq \emptyset \qquad \forall x \in [f = f(\overline{x})] \tag{14}$$

and

$$\partial f(x) \cap N(C, x) \neq \emptyset \qquad \forall x \in C \cap [f = f(\overline{x})] \tag{15}$$

are stronger than (6) and (8), respectively. From Propositions 4 and 5 we obtain the following sufficient optimality conditions in the convex case.

Proposition 9 *Assume that* $f \in \Gamma(X)$*. If either* [(14) *and* $[f < f(\overline{x})] \neq \emptyset$ *are satisfied*] *or* [*C is convex, and* (15) *and* $C \cap [f < f(\overline{x})] \neq \emptyset$ *are satisfied*]*, then* $\overline{x}$ *is a solution of* (P)*.*

Proof It is sufficient to observe that the restriction of f at any segment included in dom f is continuous because f is convex and lsc (see [21, Prop. 2.1.6]). □

Because condition (14) is weaker than condition (12) (and (15) is weaker than (13)) when $\partial f(x)$ is nonempty, from Proposition 9 we obtain the well-known characterizations of the optimality of $\overline{x}$ obtained by Strekalovski [18, 19], Hirriart-Urruty and Ledyaev [10], as well as the recent characterization of Tsevendorj [20]; in all these papers the function f is continuous on int(dom f) which contains C. Note also that Tsevendorj [20] uses for the first time conditions of type (14) instead of (12) in characterizing solutions of (P).

Another necessary optimality condition can be obtained using Ellaia–Hassouni theorem (see [5]) that states that for the locally Lipschitz function $g : D \to \mathbb{R}$ with $D \subset (X, \|\cdot\|)$ an open convex set, g is quasiconvex if and only if $\partial^0 g(x) \subset \partial^{\circledast} g(x)$ for every $x \in D$, where $\partial^0 g(x)$ is the *Clarke subdifferential* of g at x; see the book [16] by Rockafellar and Wets for the definition of the Clarke subdifferential as well as for other types of subdifferentials. Recall that f is *semistrictly quasiconvex* if $f(y) < f(x) < \infty$ implies that $f(\lambda x+(1-\lambda)y) < f(x)$ for every $\lambda \in (0,1)$. By [1, Prop. 3.30], if dom f is convex, f is lsc on dom f and f is semistrictly quasiconvex then f is quasiconvex.

Proposition 10 *Assume that X is a normed vector space, dom f is an open convex set and f is locally Lipschitz on dom f and semistrictly quasiconvex. If $[f < f(\overline{x})] \neq \emptyset$ and $\overline{x}$ is a solution of (P), then*

$$\partial^0 f(x) \subset N(C,x) \qquad \forall x \in [f = f(\overline{x})]. \tag{16}$$

Proof Because f is continuous on dom f and semistrictly quasiconvex, f is quasiconvex. By Ellaia–Hassouni theorem we have $\partial^0 f(x) \subset \partial^{\circledast} f(x)$ for every $x \in \text{dom } f$. Because $[f < f(\overline{x})] \neq \emptyset$ and f is semistrictly quasiconvex, f has no local minimizers on $[f = f(\overline{x})]$. Using Proposition 6 we get $\partial^{\circledast} f(x) = \partial^{\nu} f(x)$ for every $x \in [f = f(\overline{x})]$. The conclusion follows using Proposition 4 (i). □

As in [3, Def. 2.1], we say that the locally Lipschitz function $g : D \to \mathbb{R}$ (with $D \subset (X, \|\cdot\|)$ an open set) is ∂^0*-pseudoconvex* if $\partial^0 g(x) \subset \partial^* g(x)$ for every $x \in D$.

Proposition 11 *Assume that X is a normed vector space, dom f is an open convex set and f is ∂^0 -pseudoconvex on dom f. If $[f < f(\overline{x})] \neq \emptyset$, then "$\overline{x}$ is a solution of (P)" is equivalent to anyone of the conditions* (16) *and*

$$\partial^0 f(x) \cap N(C,x) \neq \emptyset \qquad \forall x \in [f = f(\overline{x})]. \tag{17}$$

Proof Since dom f is convex and f is ∂^0-pseudoconvex on dom f, f is semistrictly quasiconvex. So, if $\overline{x}$ is a solution of (P), by Proposition 10, (16) holds. Because $\partial^0 f(x)$ is nonempty it is clear that (16) $\Rightarrow$ (17).

Assume that (17) is verified. Since $\partial^0 f(x) \subset \partial^* f(x)$ for every $x \in \text{dom } f$, the conclusion follows using Proposition 4(ii). □

The variant for C convex of the previous result is the following; in its proof one uses Proposition 5 instead of Proposition 4.

Proposition 12 *Assume that X is a normed vector space,* dom f *is an open convex set and f is ∂^0-pseudoconvex on* dom f. *If C is convex and $C \cap [f < f(\overline{x})] \neq \emptyset$, then "$\overline{x}$ is a solution of (P)" is equivalent to anyone of the conditions*

$$\partial^0 f(x) \subset N(C,x) \qquad \forall x \in C \cap [f = f(\overline{x})], \tag{18}$$
$$\partial^0 f(x) \cap N(C,x) \neq \emptyset \qquad \forall x \in C \cap [f = f(\overline{x})].$$

The characterization (18) of the solution $\overline{x}$ of (P) in the preceding result is obtained by Dutta [3, Th. 2.1] for $X = \mathbb{R}^n = \text{dom } f$.

The results above are specific to the case when f is quasiconvex, even if this assumption is not explicitly imposed in every statement. In fact this is the explanation for the fact that in many results the convexity of C is not asked (for f quasiconvex we have $\sup_{x\in C} f(x) = \sup_{x\in \text{conv } C} f(x)$; if moreover f is lsc, $\sup_{x\in C} f(x) = \sup_{x\in \overline{\text{conv}} C} f(x)$).

3 Maximization of min-convex functions

Consider $f := \min_{i\in\overline{1,p}} f_i$ with $f_i : X \to \overline{\mathbb{R}}$ proper for $i \in \overline{1,p} := \{1,2,\ldots,p\}$. We assume that dom $f = \cap_{i\in\overline{1,p}}$ dom $f_i \neq \emptyset$, and so f is proper. As usual, for $x \in X$ we introduce the set $I(x) := \{i \in \overline{1,p} \mid f_i(x) = f(x)\}$; moreover, following Tsevendorj [20], for $C \subset \operatorname{dom} f$, $x \in C$ and $k \in \overline{1,p}$ we set

$$C_k(x) := \left\{y \in C \mid f_j(y) > f(x) \quad \forall j \in \overline{1,p} \setminus \{k\}\right\}.$$

It is clear that $x \in C_k(x)$ if and only if $I(x) = \{k\}$; moreover, $C_k(x)$ might be empty. Of course, if $\overline{x} \in C$ is a solution of (P) and $k \in I(\overline{x})$, then $f_k(x) \leq f_k(\overline{x})$ for every $x \in C_k(\overline{x})$ and so one can envisage to use a result from the preceding section.

Proposition 13 *Let f and C be as above and $\overline{x} \in C$.*

(i) *If $\overline{x}$ is a solution of (P) then*

$$\partial^\nu f_k(x) \subset N(C_k(\overline{x}), x) \quad \forall k \in I(\overline{x}),\ \forall x \in [f_k = f(\overline{x})].$$

(ii) *Assume that for some $k \in I(\overline{x})$ one has $[f_k < f_k(\overline{x})] \neq \emptyset$, $\operatorname{dom} f_k$ is convex and the restriction of f_k to any segment contained in $\operatorname{dom} f_k$ is continuous; then $\overline{x}$ is a solution of (P) provided*

$$\partial^* f_k(x) \cap N(C_k(\overline{x}), x) \neq \emptyset \quad \forall x \in [f_k = f(\overline{x})].$$

Proof (i) Fix $k \in I(\overline{x})$. As mentioned above, $\overline{x}$ is a solution of the problem: maximize $f_k(x)$ subject to $x \in \{\overline{x}\} \cup C_k(\overline{x})$. The conclusion follows applying Proposition 4(i).

(ii) Let $k \in I(\overline{x})$ be such that the mentioned conditions on f_k are verified. Applying Proposition 4(ii) for f_k and $\{\overline{x}\} \cup C_k(\overline{x})$, we obtain that $f_k(x) \leq f_k(\overline{x})$ for every $x \in C_k(\overline{x})$, and so $f(x) \leq f(\overline{x})$ for every $x \in C_k(\overline{x})$. Let now $x \in C \setminus C_k(\overline{x})$. Then there exists $j \in \overline{1,p} \setminus \{k\}$ such that $f_j(x) \leq f(\overline{x})$, whence $f(x) \leq f(\overline{x})$. Hence $\overline{x}$ is a solution of (P). □

When the functions f_i are from $\Gamma(X)$ the preceding result can be formulated as follows.

Proposition 14 *Let $f_i \in \Gamma(X)$ for $i \in \overline{1,p}$, $f := \min_{i \in \overline{1,p}} f_i$ be proper and $\overline{x} \in C \subset \operatorname{dom} f$. If $\overline{x}$ is a solution of (P) then*

$$\partial f_k(x) \subset N(C_k(\overline{x}), x) \quad \forall k \in I(\overline{x}),\ \forall x \in [f_k = f(\overline{x})].$$

Conversely, assume that for some $k \in I(\overline{x})$ one has $[f_k < f_k(\overline{x})] \neq \emptyset$ and

$$\partial f_k(x) \cap N(C_k(\overline{x}), x) \neq \emptyset \quad \forall x \in [f_k = f(\overline{x})].$$

Then $\overline{x}$ is a solution of (P).

Tsevendorj [20] obtained the preceding result for $X = \mathbb{R}^n = \operatorname{dom} f_i$ for $i \in \overline{1,p}$ and C a convex compact set.

4 Optimality conditions in the nonconvex case

In this section we consider the case when no convexity assumptions are imposed on f; however, we assume that C is convex in this case. The first optimality condition for problem (P) without asking any convexity of the objective function was established by Hirriart-Urruty and Ledyaev in [10].

The proofs of our optimality conditions are based on some simple results related to sublinear functions we state below.

Recall that $p : X \to \mathbb{R} \cup \{\infty\}$ is *sublinear* if $p(0) = 0$, $p(x + x') \leq p(x) + p(x')$ and $p(tx) = tp(x)$ for all $x, x' \in X$ and $t \in (0, \infty)$. Also recall that the *indicator function* of

$A \subset X$ is the function $\iota_A : X \to \overline{\mathbb{R}}$ defined by $\iota_A(x) := 0$ for $x \in A$ and $\iota_A := +\infty$ for $x \in X \setminus A$, the (*radial*) *tangent cone* of A at $x \in A$ is the set $T(A, x) := \mathrm{cl}(\mathbb{R}_+(A - x))$, while its *support* function is $\sigma_A := (\iota_A)^*$.

Lemma 15 *Let $p : X \to \mathbb{R}$ be a continuous sublinear function, $A \subset X$ a convex set and $x \in A$. Then*

$$[\exists u \in A : p(u - x) < 0] \Leftrightarrow 0 \notin \partial p(0) + N(A, x) \tag{19}$$

and

$$\partial p(0) \subset N(A, x) \Leftrightarrow [p(u) \le 0 \quad \forall u \in T(A, x)] \Leftrightarrow [p(u - x) \le 0 \quad \forall u \in A]. \tag{20}$$

Proof Observe that "$p(u - x) \ge 0$ for every $u \in A$" means that x is a global minimum of $p(\cdot - x) + \iota_A$, which is equivalent to $0 \in \partial(p(\cdot - x) + \iota_A)(x)$. Since $p(\cdot - x)$ is continuous, the latter condition is equivalent to $0 \in \partial p(0) + \partial \iota_A(x) = \partial p(0) + N(A, x)$. Then (19) follows by negation.

It is obvious that $p(u - x) \le 0$ for every $u \in A$ if and only if $p(u) \le 0$ for every $u \in \mathbb{R}_+(A - x)$. We get now the last equivalence in (20) using the continuity of p. For the first just observe that

$$\partial p(0) \subset N(A, x) \Leftrightarrow p = \sigma_{\partial p(0)} \le \sigma_{N(A,x)} = \iota_{T(A,x)} \Leftrightarrow [p(u) \le 0 \quad \forall u \in T(A, x)].$$

The proof is complete. □

As seen from the proof of the preceding lemma, the equivalence (19) is derived from the optimality conditions of a convex programming problem. On the contrary, the equivalence of the first and the last propositions in (20) is related to problem (P); indeed, taking $f := p$ and $C := A - x$, we have that 0 is a solution of (P) if and only if $\partial f(0) \subset N(C, 0)$.

Lemma 16 *Let $X_0 \subset X$ be a linear space and $p : X \to \mathbb{R} \cup \{\infty\}$ a sublinear function. Then*

$$[p(u) \le 0 \quad \forall u \in X_0] \Leftrightarrow [p(u) = 0 \quad \forall u \in X_0]. \tag{21}$$

Proof As the implication $\Leftarrow$ is obvious, let us assume that $p(u) \le 0$ for every $u \in X_0$ and that $p(\overline{u}) < 0$ for some $\overline{u} \in X_0$. Then $0 = p(0) = p(\overline{u} + (-\overline{u})) \le p(\overline{u}) + p(-\overline{u}) < 0$, a contradiction. □

Before establishing optimality conditions in the nonconvex case we recall the *Dini lower* and *upper directional derivatives* of f at $x \in \mathrm{dom}\, f$:

$$f_l'(x, u) := \liminf_{t \downarrow 0} \frac{f(x + tu) - f(x)}{t}, \quad f_u'(x, u) := \limsup_{t \downarrow 0} \frac{f(x + tu) - f(x)}{t} \quad (u \in X).$$

Proposition 17 *Assume that for every $x \in C \cap [f = f(\overline{x})]$ there exists $\partial_u f(x) \subset X^*$ a nonempty convex w^*-compact set such that $f_u'(x, \cdot) \ge \sigma_{\partial_u f(x)}$. If $\overline{x}$ is a solution of (P) then*

$$\partial_u f(x) \subset N(C, x) \qquad \forall x \in C \cap [f = f(\overline{x})]. \tag{22}$$

Proof Taking $p_x := \sigma_{\partial_u f(x)}$, p_x is a continuous sublinear function and $\partial p_x(0) = \partial_u f(x)$. Let $y \in C$ and $x \in C \cap [f = f(\overline{x})]$. Then $(1 - t)x + ty \in C$, and so $f(x + t(y - x)) \le f(\overline{x}) = f(x)$ for every $t \in (0, 1)$. It follows that

$$p_x(y - x) \le f_u'(x, y - x) = \limsup_{t \downarrow 0} \frac{f(x + t(y - x)) - f(x)}{t} \le 0.$$

By Lemma 15 we obtain that $\partial p_x(0) = \partial_u f(x) \subset N(C, x)$. Hence (22) holds. □

Before providing a sufficient optimality condition which corresponds to the previous result, let us recall that the *quasi relative interior* of the convex set $A \subset X$ is the set

$$\text{qri } A := \{x \in A \mid T(A, x) \text{ is a linear space }\}.$$

It is known that the intrinsic core ${}^iA$ is included in qri A (${}^iA =$ qri A if dim $X < \infty$) and $(1-\lambda)x_0 + \lambda x \in$ qri A for all $x_0 \in$ qri A, $x \in A$ and $\lambda \in [0, 1)$; in particular $A \subset$ cl(qri A) when qri $A \neq \emptyset$ (see Borwein and Lewis [2] and [21, Prop. 1.2.7]). Note also that the quasi relative interior of a nonempty closed convex subset of a separable Banach space is nonempty (see [2, Th. 2.19] or [21, Prop. 1.2.9]).

Proposition 18 *Assume that f is continuous on C and that for every $x \in C \cap [f = f(\overline{x})]$ there exists $\partial_l f(x) \subset X^*$ a nonempty convex w^*-compact set such that $f_l'(x, \cdot) \leq \sigma_{\partial_l f(x)}$ and*

$$0 \notin \partial_l f(x) + N(C, x) \qquad \forall x \in C \cap [f = f(\overline{x})]. \tag{23}$$

If qri $C \neq \emptyset$ *and*

$$\partial_l f(x) \subset N(C, x) \qquad \forall x \in C \cap [f = f(\overline{x})], \tag{24}$$

then $\overline{x}$ is a solution of (P).

Proof Taking $p_x := \sigma_{\partial_l f(x)}$, p_x is a continuous sublinear function and $\partial p_x(0) = \partial_l f(x)$. By (19), for every $x \in C \cap [f = f(\overline{x})]$, there exists $u \in C$ such that $p_x(u - x) < 0$, and so $f_l'(x, u - x) := \liminf_{t \downarrow 0} t^{-1}[f(x + t(u - x)) - f(x)] < 0$. It follows that there exists $t \in (0, 1)$ such that $f(x_0) < f(x) = f(\overline{x})$, where $x_0 := (1 - t)x + tu \in C$. Hence $[f < f(\overline{x})] \neq \emptyset$. Because $C \subset$ cl(qri C) and f is continuous on C, there exists even $x_0 \in$ qri C such that $f(x_0) < f(\overline{x})$. Assume that $\overline{x}$ is not a solution of (P). Then there exists some $x_1 \in C$ such that $f(x_1) > f(\overline{x})$. Because $f|_{[x_0,x_1]}$ is continuous, it follows that there exists some $\lambda \in (0, 1)$ such that $f(x) = f(\overline{x})$, where $x := (1-\lambda)x_0 + \lambda x_1$. Since $x_1 \in C$ and $x_0 \in$ qri C, $x \in$ qri C, too. From (24), we get $\partial_l f(x) \subset N(C, x)$, and so, by (20), $p_x(u) \leq 0$ for every $u \in T(C, x)$. Since $x \in$ qri C, $T(C, x)$ is a linear space; by (21) we have $p_x(u - x) = 0$ for every $u \in C$, contradicting (23). The proof is complete. □

The preceding result extends significantly Dutta's result [3, Th. 3.1] which is stated for $X = \mathbb{R}^n =$ dom f and f a locally Lipschitz function with the property that for each $x \in X$ there exists a bounded convexificator which is, as multifunction, upper semicontinuous. Note that Propositions 17 and 18, put together, extend Theorem 2.1 of Hirriart-Urruty and Ledyaev [10], which is stated for X a Hilbert space, for f a regular locally Lipschitz function and for C a closed convex set; just take $\partial_l f(x) := \partial_u f(x) := \partial^0 f(x)$. However, we do not cover the case when X is not separable because we do not know if qri C is nonempty in this case. Also note that our proof is simpler and very different from those in [3] and [10].

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